

Solutions to Mathematics Workbook-1 | Quadratic Equations & Inequalities [2022]

JEE (Advanced)

Daily Tutorial Sheet 1

1.(D) PLAN If $ax^2 + bx + c = 0$ has roots α and β , then $\alpha + \beta = -b/a$ and $\alpha\beta = \frac{c}{a}$. Find the values of $\alpha + \beta$ and

$\alpha\beta$ and then put in $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta$ to get required value.

Given, α and β are roots of $px^2 + qx + r = 0$, $p \neq 0$.

$$\therefore \alpha + \beta = \frac{-q}{p}, \quad \alpha\beta = \frac{r}{p} \quad \dots (i)$$

Since, p , q and r are in AP.

$$\therefore 2q = p + r \quad \dots (ii)$$

$$\text{Also, } \frac{1}{\alpha} + \frac{1}{\beta} = 4 \Rightarrow \frac{\alpha + \beta}{\alpha\beta} = 4$$

$$\Rightarrow \alpha + \beta = 4\alpha\beta \Rightarrow \frac{-q}{p} = \frac{4r}{p} \quad [\text{from Eq. (i)}]$$

$$\Rightarrow q = -4r$$

On putting the value of q in Eq. (ii), we get $\Rightarrow 2(-4r) = p + r \Rightarrow p = -9r$

$$\text{Now, } \alpha + \beta = \frac{-q}{p} = \frac{4r}{-9r} = -\frac{4}{9} \text{ and } \alpha\beta = \frac{r}{p} = \frac{r}{-9r} = -\frac{1}{9} \therefore (\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = \frac{16}{81} + \frac{4}{9} = \frac{16 + 36}{81}$$

$$\Rightarrow (\alpha - \beta)^2 = \frac{52}{81} \Rightarrow |\alpha - \beta| = \frac{2}{9}\sqrt{13}$$

2.(B) Sum of roots $= \frac{\alpha^2 + \beta^2}{\alpha\beta}$ and product $= 1$

$$\text{Given, } \alpha + \beta = -p \text{ and } \alpha^3 + \beta^3 = q \Rightarrow (\alpha + \beta)(\alpha^2 - \alpha\beta + \beta^2) = q$$

$$\therefore \alpha^2 + \beta^2 - \alpha\beta = \frac{-q}{p} \quad \dots (i)$$

$$\text{and } (\alpha + \beta)^2 = p^2$$

$$\Rightarrow \alpha^2 + \beta^2 + 2\alpha\beta = p^2 \quad \dots (ii)$$

$$\text{From Eqs. (i) and (ii), we get } \alpha^2 + \beta^2 = \frac{p^3 - 2q}{3p} \text{ and } \alpha\beta = \frac{p^3 + q}{3p}$$

$$\therefore \text{Required equation is, } x^2 - \frac{(p^3 - 2q)x}{(p^3 + q)} + 1 = 0$$

$$\Rightarrow (p^3 + q)x^2 - (p^3 - 2q)x + (p^3 + q) = 0$$

3.(A) Since, roots are real, therefore $D \geq 0 \Rightarrow 4(a + b + c)^2 - 12\lambda(ab + bc + ca) \geq 0$

$$\Rightarrow (a + b + c)^2 \geq 3\lambda(ab + bc + ca) \Rightarrow a^2 + b^2 + c^2 \geq (ab + bc + ca)(3\lambda - 2)$$

$$\Rightarrow 3\lambda - 2 \leq \frac{a^2 + b^2 + c^2}{ab + bc + ca} \quad \dots (i)$$

Also, $\cos A = \frac{b^2 + c^2 - a^2}{2bc} < 1 \Rightarrow b^2 + c^2 - a^2 < 2bc$

Similarly, $c^2 + a^2 - b^2 < 2ca$ and $a^2 + b^2 - c^2 < 2ab$

$$\Rightarrow a^2 + b^2 + c^2 < 2(ab + bc + ca) \Rightarrow \frac{a^2 + b^2 + c^2}{ab + bc + ca} < 2 \quad \dots (ii)$$

From Eqs. (i) and (ii), we get $3\lambda - 2 < 2 \Rightarrow \lambda < \frac{4}{3}$

4.(A) Since, α, β are the roots of $x^2 + px + q = 0$ and α^4, β^4 are the roots of $x^2 - rx + s = 0$.

$$\Rightarrow \alpha + \beta = -p; \alpha\beta = q; \alpha^4 + \beta^4 = r \text{ and } \alpha^4\beta^4 = s$$

Let roots of $x^2 - 4qx + (2q^2 - r) = 0$ be α' and β'

Now, $\alpha'\beta' = (2q^2 - r) = 2(\alpha\beta)^2 - (\alpha^4 + \beta^4) = -(\alpha^4 + \beta^4 - 2\alpha^2\beta^2) = -(\alpha^2 - \beta^2)^2 < 0$

$\Rightarrow$ Roots are real and of opposite sign.

5.(D) Case 1: $x^2 + 4x - 60 = 0$

$$(x+10)(x-6) \Rightarrow x = 6, -10$$

Case 2: $x^2 - 5x + 5 = 1$

$$\Rightarrow x^2 - 5x + 4 = 0 \Rightarrow x = 1, 4$$

Case 3: $x^2 - 5x + 5 = -1$ and $x^2 + 4x - 60$ is even integers.

$$\Rightarrow x^2 - 5x + 6 = 0 \Rightarrow x = 2, 3$$

For $x = 2 \Rightarrow x^2 + 4x - 60$ is even

For $x = 3 \Rightarrow x^2 + 4x - 60$ is odd

$$\therefore x = 2$$

Sum of roots = $6 - 10 + 1 + 4 + 2 = 3$

6. $x^2 - 10ax - 11b = 0 - (c, d)$

$$x^2 - 10cx - 11d = 0 - (a, b)$$

$$a + b = 10c, c + d = 10a$$

$$\therefore a + b + c + d = 10(a + c)$$

$$\text{and } b - d = 11(c - a)$$

$$\text{Also } c^2 - 10ac - 11b = 0 \quad \dots (i)$$

$$a^2 - 10ac - 11d = 0 \quad \dots (ii)$$

$$(i) - (ii) \quad c^2 - a^2 = 11(b - d) \Rightarrow (c - a)(c + a) = 11(b - d) = 11 \times 11 \times (c - a)$$

$$\therefore c + a = 11 \times 11 \Rightarrow a + b + c + d = 11 \times 11 \times 10 = 1210$$

7. Since, $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ and $\alpha + \delta + \beta + \delta = \frac{B}{A}$, $(\alpha + \delta)(\beta + \delta) = \frac{C}{A}$

Now, $\alpha - \beta = (\alpha + \delta) - (\beta + \delta)$

$$\Rightarrow (\alpha - \beta)^2 = [(\alpha + \delta) - (\beta + \delta)]^2 \Rightarrow (\alpha + \beta)^2 - 4\alpha\beta = \overline{(\alpha + \delta + \beta + \delta)}^2 - 4(\alpha + \delta)(\beta + \delta)$$

$$\Rightarrow \left(-\frac{b}{a}\right)^2 - \frac{4c}{a} = \left(-\frac{B}{A}\right)^2 - \frac{4C}{A} \Rightarrow \frac{b^2}{a^2} - \frac{4c}{a} = \frac{B^2}{A^2} - \frac{4C}{A} \Rightarrow \frac{b^2 - 4ac}{a^2} = \frac{B^2 - 4AC}{A^2}$$

8. Let α, β are roots of $ax^2 + bx + c = 0$ Given, $\alpha = \beta^n \Rightarrow \alpha\beta = \frac{c}{a} \Rightarrow \beta^{n+1} = \frac{c}{a} \Rightarrow \beta = \left(\frac{c}{a}\right)^{1/(n+1)}$

It must satisfy $ax^2 + bx + c = 0$

i.e. $a\left(\frac{c}{a}\right)^{2/(n+1)} + b\left(\frac{c}{a}\right)^{1/(n+1)} + c = 0$

$$\Rightarrow \frac{a \cdot c^{2/(n+1)}}{a^{2/(n+1)}} + \frac{b \cdot c^{1/(n+1)}}{a^{1/(n+1)}} + c = 0 \Rightarrow \frac{c^{1/(n+1)}}{a^{1/(n+1)}} \left\{ \frac{a \cdot c^{1/(n+1)}}{a^{1/(n+1)}} + b + \frac{c \cdot a^{1/(n+1)}}{c^{1/(n+1)}} \right\} = 0$$

$$\Rightarrow a^{n/(n+1)} c^{1/(n+1)} + b + c^{n/(n+1)} a^{1/(n+1)} = 0 \Rightarrow (a^n c)^{1/(n+1)} + (c^n a)^{1/(n+1)} + b = 0$$

9. Since, α, β are the roots of $x^2 + px + q = 0$ and γ, δ are the roots of $x^2 + rx + s = 0$

$\therefore \alpha + \beta = -p, \alpha\beta = q$

and $\gamma + \delta = -r, \gamma\delta = s$

Now, $(\alpha - \gamma)(\alpha - \delta)(\beta - \gamma)(\beta - \delta) = [\alpha^2 - (\gamma + \delta)\alpha + \gamma\delta][\beta^2 - (\gamma + \delta)\beta + \gamma\delta] = (\alpha^2 + r\alpha + s)(\beta^2 + r\beta + s)$
 $= (\alpha\beta)^2 + r(\alpha + \beta)\alpha\beta + s(\alpha^2 + \beta^2) + \alpha\beta r^2 + rs(\alpha + \beta) + s^2 = q^2 - rqp + s(p^2 - 2q) + qr^2 - rsp + s^2$
 $= (q - s)^2 - rqp - rsp + sp^2 + qr^2$

10. Since, $\alpha + \beta = -p, \alpha\beta = 1$ and $\gamma + \delta = -q, \gamma\delta = 1$

Now, $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = \{\alpha\beta - \gamma(\alpha + \beta) + \gamma^2\}\{\alpha\beta + \delta(\alpha + \beta) + \delta^2\} = \{1 - \gamma(-p) + \gamma^2\}\{1 + \delta(-p) + \delta^2\}$
 $= (1 + \gamma^2 + \gamma p)(1 - \delta p + \delta^2) = (-q\gamma + \gamma p)(-\delta p - \delta q) \quad [\because \gamma^2 + q\gamma + 1 = 0 \text{ and } \delta^2 + q\delta + 1 = 0]$
 $= (q^2 - p^2)(\gamma\delta) = q^2 - p^2 \quad [\because \gamma\delta = 1]$